Short Answer Questions

Please answer all questions

1. a) What's the state space of a classical bit? (1 mark)
   b) What's the state space of a qubit? (1 mark)
   c) How many classical bits can be stored in a qubit? Briefly justify your answer (3 marks).

2. a) What is a pure state? (1 mark)
   b) What is the purity of a state? (1 mark)
   c) Are the following states pure or mixed?
   (i) $\rho = \frac{1}{2}(\mathbb{I} + \sigma_x)$
   (ii) $\rho = \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix}$
   (iii) $\rho = \frac{1}{4}|\Phi_+\rangle\langle\Phi_+| + \frac{1}{4}|\Phi_-\rangle\langle\Phi_-| + \frac{3}{4}|00\rangle\langle 00| - \frac{1}{4}|11\rangle\langle 11|$
   (3 marks)

3. a) Argue geometrically that any pure state has a Bloch vector of norm 1 and hence can be written as $|\psi\rangle\langle\psi|$?
   (3 marks)
   b) What is the effect of evolving the state $\rho = 3/4|+\rangle\langle+| + 1/4|-\rangle\langle-|$ for $t = \pi/4$ under $H = \sigma_y$? State the final state and sketch this evolution on the Bloch sphere.
   (4 marks)

4. a) What is meant by the purification of a state ρ? (2 mark)
   b) State a purification for the state $\rho = 1/4|0\rangle\langle 0| + 1/2|-\rangle\langle-| + 1/4|1\rangle\langle 1|$ using 3 qubits. (1 mark)
   c) How could you purify ρ using only 2 qubits? (4 marks)

5. What is a POVM? How does it differ from a Hermitian measurement? (6 marks)

6. Use the unitary invariance of the Schatten p-norms and the triangle inequality to show that

$$||U^N - V^N||_p \leqslant N||U - V||_p \,.$$

   (6 marks)

7. a) Define the vectorisation of an operator X. (1 mark)

   b) Show that $M \otimes \mathbb{I}|\Omega\rangle = \mathbb{I} \otimes M^T|\Omega\rangle$ where $|\Omega\rangle$ is the unnormalized maximally entangled state $\sum_i |ii\rangle$. (3 marks)

   c) Hence show that $\mathrm{Tr}(AB) = \langle\Omega|A \otimes B^T|\Omega\rangle$ (3 marks)

8. a) Carefully state the free operations of the resource theory of entanglement. (6 marks)

   b) What are the free states corresponding to these free operations? (1 mark)

Long Answer Questions

Please pick 2 questions to answer

Question A - Quantum Channels

Consider the depolarising channel

$$\mathcal{E}(\rho) = p\frac{\mathbb{I}}{d} + (1-p)\rho\,. \tag{1}$$

a) Show that $\mathrm{Tr}[\mathcal{E}(\rho)^2] \leqslant \mathrm{Tr}[\rho^2]$. (2 marks)

b) Sketch/describe the action of $\mathcal{E}$ for a single qubit on the Block sphere. (3 marks)

c) State what is meant by the Kraus form of a quantum channel. (2 marks)

d) Find a set of Kraus operators to represent the action of $\mathcal{E}$ for a single qubit state. (7 marks)

e) What Kraus operators (non-normalised is fine) could be used to represent $\mathcal{E}$ for an arbitrary dimensional system? (2 marks)

Consider instead the single qubit quantum operation

$$\mathcal{E}(\rho) = q\mathrm{Tr}[\rho]\frac{\mathbb{I}}{2} + p\rho + (1-q-p)\rho^T \tag{2}$$

f) Is this a genuine quantum channel? Be precise! (9 marks)

Question B - Entropy

a) Use Jensen's inequality to show that the classical relative entropy $S(P||Q)$ is always non-negative. (5 marks)

For any doubly stochastic matrix P, concave function f and probability distribution $\boldsymbol{p}$

$$\sum_j P_{ij} f(p_j) \leqslant f\left(\sum_j P_{ij} p_j\right) . \tag{3}$$

b) Use this to show that the quantum relative entropy $S(\rho||\sigma)$ is always non-negative. (9 marks).

c) *Hence* prove that the Von Neumann entropy is subadditive

$$S(\rho) \leqslant S(\rho_A) + S(\rho_B) . \tag{4}$$

(4 marks)

d) The collision of two previously uncorrelated particles A and B can be modelled as

$$\rho_A \otimes \rho_B \to U(\rho_A \otimes \rho_B)U^\dagger . \tag{5}$$

What is the effect of the collision on the entropy of i. the joint system and ii. the separate particles A and B?

(3 marks)

e) What about if we performed the same analysis running the process backwards? Is this mysterious? (4 marks)

Question C - Entanglement Catalysts

a) State Nielsen's Majorization Theorem. (1 marks)

b) Show that any probability distribution is majorized by the distribution $p(x) = 1$ for $x = x_0$ and $p(x) = 0$ for $x \neq x_0$. (2 marks)

c) What does this imply about the states that can be deterministically transformed to or from the product state of two pure states $|\psi\rangle_A \otimes |\phi\rangle_B$ via LOCC? (2 marks)

d) Prove that if $\boldsymbol{a} \prec \boldsymbol{c}$ and $\boldsymbol{b} \prec \boldsymbol{d}$ then $\boldsymbol{a} \otimes \boldsymbol{b} \prec \boldsymbol{c} \otimes \boldsymbol{d}$.

(Hint - there are other ways of doing this but you could start by showing $\boldsymbol{a} \otimes \boldsymbol{b} \prec \boldsymbol{c} \otimes \boldsymbol{b}$.)

(5 marks)

e) Interpret this mathematical result physically in terms of LOCC on pure bipartite quantum systems. (1 marks)

f) Show that

$$|\psi\rangle = \sqrt{\frac{2}{5}}(|00\rangle + |11\rangle) + \sqrt{\frac{1}{10}}(|22\rangle + |33\rangle) \tag{6}$$

cannot be transformed deterministically to

$$|\phi\rangle = \sqrt{\frac{1}{2}}|00\rangle + \frac{1}{2}(|11\rangle + |22\rangle) \tag{7}$$

via LOCC, but $|\psi\rangle \otimes |\chi\rangle$ can be transformed to $|\phi\rangle \otimes |\chi\rangle$ where

$$|\chi\rangle = \sqrt{\frac{3}{5}}|00\rangle + \sqrt{\frac{2}{5}}|11\rangle\,. \tag{8}$$

(4 marks)

g) Why is $|\chi\rangle$ sometimes called an entanglement catalyst? (1 marks)

e) In the limit of large n how many copies of $|\phi\rangle$ can be generated from $|\psi\rangle$ probabilistically using LOCC? That is, find the value m such that $|\psi\rangle^{\otimes n} \to |\phi\rangle^{\otimes m}$ via LOCC (with high probability). For concreteness take $n = 10^6$. (4 marks)

f) Describe (at a high level) a protocol that could be used to implement this probabilistic transformation. (5 marks)

Question D - Shot Noise

We are interested in computing the expectation value of the Hamiltonian H of the transverse field Ising model for two spins

$$H = J\left(Z \otimes Z\right) + g\left(X \otimes \mathbb{1} + \mathbb{1} \otimes X\right), \tag{9}$$

where J and g are constants (interaction and field strength respectively). The spectrum of this Hamiltonian is given by $\lambda(H) = \{\pm J, \pm\sqrt{J^2 + 4g^2}\}$.

Assume that we are able to create product states i.e. $|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ where $|\psi_i\rangle = \frac{1}{2}(\mathbb{I} + \boldsymbol{r^{(i)}}.\boldsymbol{\sigma})$ for $(i = 1, 2)$ can be any single qubit pure state.

a) Use Chebyshev's bound to deduce as a function of $g, J, \boldsymbol{r^{(1)}}$ and $\boldsymbol{r^{(2)}}$ the number of shots required to estimate H up to a precision ϵ with probability at least $1 - \delta$. (9 marks)

b) Repeat the same calculation this time using Hoeffding's Inequality. (5 marks)

We next want to determine which product states experience the minimal shot noise.

c) First argue why we can reduce our optimization domain to a circle on the Bloch sphere instead of the entire Bloch sphere. (1 mark)

d) Considering the limit $g << J$ show that you can even restrict your optimization domain to a single axis of the Bloch sphere i.e. to just two pure states. What are these states? (1 mark)

e) Show that the variance of your estimator in this case is given by $2g^2/N$. (5 marks)

f) Now suppose that you are able to create entangled states. Compute the effect of shot noise for the Bell state $|\phi_-\rangle$. (4 marks)